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METHODS FOR CONSTRUCTION OF ODD NUMBER POINTED POLYGONS

Abstract: *The purpose of this paper is to present methods for constructing of polygons with an odd number of sides, although some of them may not be built only with compass and straightedge. Odd pointed polygons are difficult to construct accurately, though there are relatively simple ways of making a good approximation which are accurate within the normal working tolerances of design practitioners. The paper illustrates rather complicated constructions to produce unconstructible polygons with an odd number of sides, constructions that are particularly helpful for engineers, architects and designers. All methods presented in this paper provide practice in geometric representations.*

Key words: *regular polygon, pentagon, heptagon, nonagon, hendecagon, pentadecagon, heptadecagon.*

1. INTRODUCTION

For a specialist inured to engineering graphics, plane and solid geometries exert a special fascination. Relying on theorems and relations between linear and angular sizes, geometry develops students' spatial imagination so necessary for understanding other graphic discipline, descriptive geometry, underlying representations in engineering graphics.

Construction of regular polygons with odd number of sides, just by straightedge and compass, has preoccupied many mathematicians who have found ingenious solutions for their representation. Determination equally spaced points around the circumference of a circle enable various bi-dimensional models, the most prominent being those used in the construction of patterns for the purpose of decoration [1].

Carl Friedrich Gauss developed the theory of Gaussian periods (1801) that allowed him to establish geometric and algebraic criteria for the construction by compass and straightedge of a regular polygon inscribed in a circle, finding, at the age of 19, the construction of a regular polygon with 17 sides [2].

Gauss' criterion of constructible regular polygons has the following enunciation: "A regular *n*-polygon can be constructed by ruler and compass if and only if *n* is the product of a power of 2 and any number of distinct Fermat primes" [8]. A Fermat prime is a prime number of the form: $2^{\left(2^n\right)}+1$.

Therefore, an *n*-polygon is constructible if n = 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, ... while an *n*-polygon is not constructible if n = 7, 9, 11, 13, 14, 18, 19, 21, 22, 23, 25, ...

In the following paragraphs are presented constructions of polygons with odd number of sides, for which is given a general method of representation, regardless of the odd number of sides.

For any regular polygon inscribed in a circle these relationships are valid [3]:

$$L_n = 2\,R \sin\frac{\pi}{n}; \quad a_n = R\cos\frac{\pi}{n} \qquad (1)$$

where:

n – number of sides;
L_n – length of a side;
a_n – apothem (radius of inscribed circle);
R – radius of circumscribed circle.

2. THREE - POINT GEOMETRY

The construction for three point geometry is shown in fig. 1. Given a circle with center O, draw the diameter AD. From the point D as center and, with a radius in compass equal to the radius of circle, describe an arc that intersects the circle at points B and C. Points A, B and C divide the circle into three equal parts and joining these points is obtained equilateral triangle inscribed within the circle.

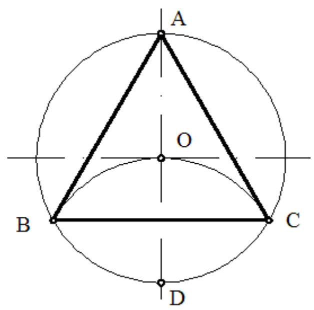


Fig. 1 Three-point geometry

We can substitute for *n=3* from the Eq. (1) to obtain:

$$L_3 = R\sqrt{3}; \quad a_3 = \frac{R}{2} \qquad (2)$$

3. CONSTRUCTION OF A PENTAGON

There are many construction methods for five point geometries [4]. One of the simpler ones is illustrated in fig. 2.

To divide a circle into five equal parts describe two perpendicular diameters. From the right end of the horizontal diameter, with a radius in compass equal to the radius of circle, draw an arc that intersects the circle in points 1 and 2. The segment 12 divides radius into two equal parts and point 4 is obtained.

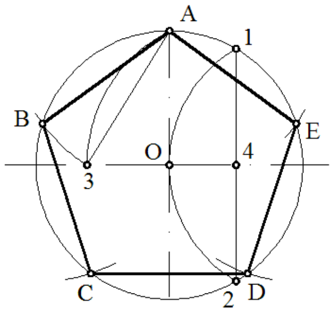


Fig. 2 Construction of a pentagon

With center 4 and radius equal to the distance 4-A draw an arc that intersects the horizontal diameter in point 3. Distance A-3 is one side of regular pentagon inscribed in the circle. Repeating this segment (as a chord) around the circle, it establish the remaining points on the circle, to get points B, C, D and E. Connecting these points with straight lines, result the regular pentagon. Distance O3 is one side of the decagon and can therefore be used to divide the circle into 10 equal parts.

An alternative construction of a pentagon is illustrated in fig. 3. In order to construct a regular pentagon we have to run through the following steps:

1. Given the length of side AB (fig. 3), with radius AB and a centre established at each end of the line, draw two circles that intersect each other at two points. Draw a vertical line between these two points.
2. Draw an arc with point 1 (the lower of the two points of intersection) as center and radius 1A which will intersect with the first two circles twice each (points 2 and 3).
3. Draw two lines from these points, extending them through the point of intersection of the last arc and the vertical line. These lines will intersect with the first two circles to locate points C and E.
4. Connect the initial points A and B with these two points of intersection to create the next two sides of the pentagon (AE and BC).
5. Draw two arcs with points C and E as centers to obtain the final point D necessary to complete the five sides of the pentagon.

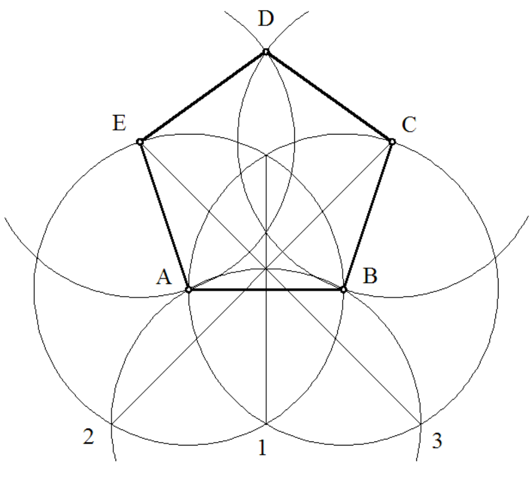


Fig. 3 Alternative construction of a pentagon

We can substitute for $n=5$ from the Eq. (1) to obtain:

$$L_5 = \frac{R\sqrt{10 - 2\sqrt{5}}}{2} \tag{3}$$

$$a_5 = \frac{R\left(\sqrt{5} + 1\right)}{4} \tag{4}$$

From the five point geometry, ten point geometries are easily developed and form the basis for many of the more attractive patterns in mosaic pavements [5].

In a regular pentagon, the ratio of the diagonal to the side is equal to the Golden Ratio:

$$\varphi = 1.6180339887... \tag{5}$$

The triangle in the middle of figure 4, with a ratio of side to base of φ, is known as a Golden Triangle. The two triangles on the sides, with a ratio of side to base of $1/\varphi$, are called Golden Gnomons. The 36°–72°–72° triangle occurs in both the pentagram and the decagon [6].

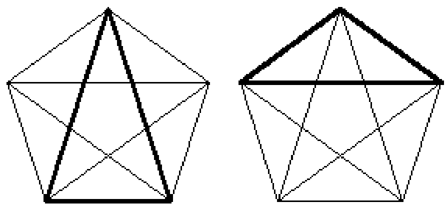


Fig. 4 Golden Triangle (left) and Golden Gnomon (right)

4. GEOMETRIC CONSTRUCTION OF THE POLYGON WITH SEVEN SIDES

Seven-point geometry is difficult to construct accurately – the angle at centre of a circle divided by seven is 51.4285714° – though there is a relatively simple way of making a good approximation.

The sequence of steps referring to the determination of seven pointed polygon has the following development:

1. Draw two perpendicular diameters.
2. With its centre at the intersection of these diameters draw a circle.
3. With a centre based on the lower point of vertical diameter and the same radius, draw an arc that intersects the circle in two points. One of them is point M. Connect these points of intersection (fig. 5).
4. From point F (an extremity of horizontal diameter) draw a line which meets the intersection of the vertical diameter and circle. This line will be at 45° to the horizontal diameter.
5. With M as center describe a circle with radius equal to the distance from point M to the point where the line at 45° cuts the horizontal line determined in step 3.
6. Where this circle cuts the initial circle, as at G, draw the line GF. Distance FG is the length of one of the sides of the heptagon. The additional points of the heptagon can be located by describing circles with radius the length of the segment FG.

If the radius of the circumscribed circle is $R = 20$, then the side length of the heptagon is L_7 = 17.3553-see Eq. (1). Using this method of construction we get FG = 17.2197 that defines an approximation of 99.2186%.

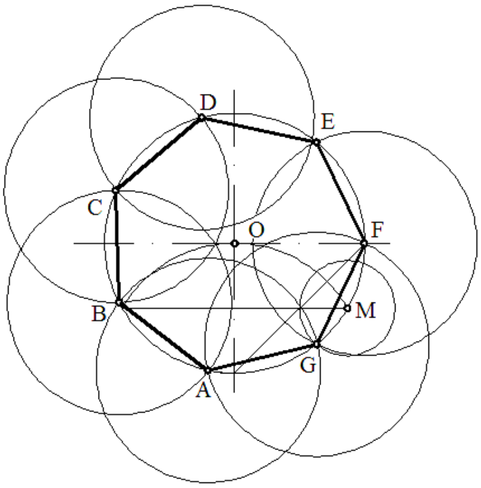


Fig. 5 Seven-point geometry

There is an alternative construction which again gives a good approximation of a heptagon (fig. 6).

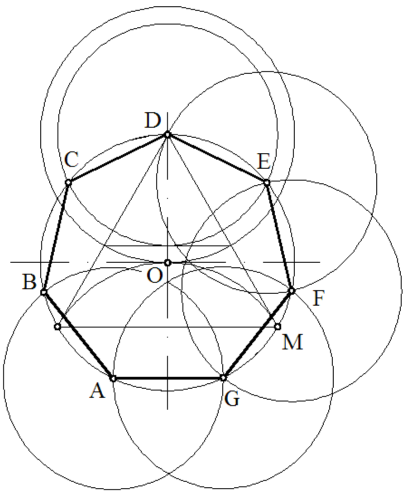


Fig. 6 Alternative construction of a heptagon

We must go through the following steps:

1. Draw two perpendicular diameters and then draw a circle with radius OD (point O is the intersection of these diameters).
2. Using the same construction illustrated in fig. 1, describe an equilateral triangle inscribed within the circle.
3. With D as center describe another circle with the same radius as the first circle. Where the two sides of the triangle cut the top peripheral circle draw a line joining them to form a smaller equilateral triangle.
4. With the compass centred on point D, draw a circle which has the bottom of the smaller triangle as its tangent. This smaller circle will cut the initial circle in two points (C and E). Distance DC (or DE) is one side of the heptagon.
5. With DE as radius, continue to draw circles centred on the circumference of the original circle. This will divide the circumference approximately into seven.

Using this method of construction we get DE = 17.3205 that defines an approximation of 99.7994%.

Figure 7 illustrates another construction of a heptagon. For dividing the circle into 7 equal parts draw two perpendicular diameters. From the lower end of the vertical diameter, with a radius in compass equal to the radius of the circle, describe an arc intersecting the circle at two points. One of them is point B. The segment that connects them cut the vertical diameter at point M. With B as center and radius BM, draw an arc to cut the circumference at A. Distance BA is one side of the heptagon. With radius BA as a chord, locate the remaining points on the circle. By connecting these points with straight lines we get the heptagon inscribed within the circle (fig. 7). If the radius is $R = 20$, *t*his construction defines an approximation of 99.8%.

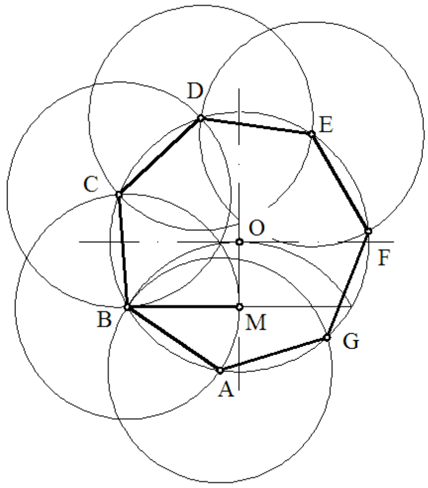


Fig. 7 Geometric **c**onstruction of the heptagon

5. GEOMETRIC CONSTRUCTION OF THE POLYGON WITH NINE SIDES

It is not possible to construct an accurate nine-pointed polygon. However, there is a method for making a good approximation.

For dividing the circle into nine equal parts draw two perpendicular diameters, with one end at point A (for vertical diameter) and the other in point 6 (for horizontal diameter). Radius O6 is divided into six equal parts. From point 6 as center with a radius equal to the distance 6-1 describe an arc that intersects the given circle in point B. With AB as radius and B as center, cut the circumference at C. With the same radius and C as center, cut the circumference at D. Repeat at E, F, G, H and I. By connecting these points with straight lines we get the nonagon inscribed within the circle (fig. 8).

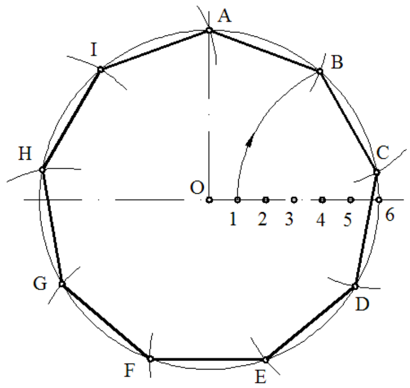


Fig. 8 Construction of a nonagon

If the radius of the circumscribed circle is $R = 30$, then the side length of the nonagon is $L_9 = 20.5212$. Using this method of construction we get AB = 20.6257 which is very close to the L_9 (differing from it by less than 0.51 percent).

The next figure illustrates an alternative construction of a nonagon. The steps are listed below:

1. Draw two perpendicular diameters that intersect at point O.
2. With radius OA and O as center draw a circle.
3. With centers established at each end of the vertical diameter construct three-point geometry within the circle (as described in fig. 1) to produce two equilateral triangles. These triangles determine a six-pointed star.
4. From the point where vertical diameter meets the horizontal line of one of the triangles draw a circle whose radius is half that of the original circle.
5. With the same radius draw another circle with the centre at the intersection of the vertical diameter and original circle, as at A.
6. Connect the points where these two similar circles meet with a horizontal line. This line will cut two of the sides of one of the equilateral triangles which form half of the six-pointed star (fig. 9).
7. From these two points of intersection, draw lines to the two points where the other equilateral triangle meets the horizontal side of the first equilateral triangle, and extend them to cut the original circle at E and F.

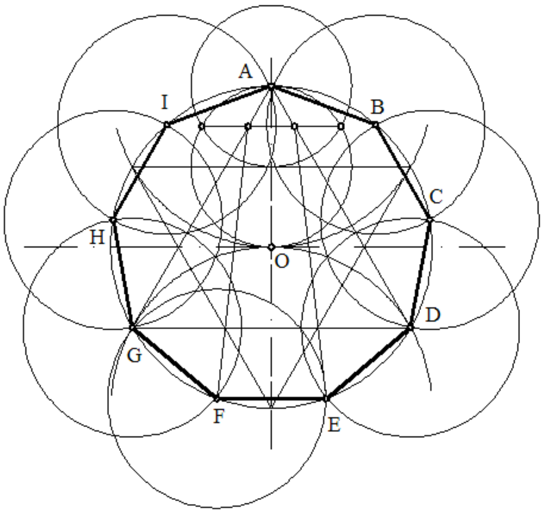


Fig. 9 Alternative construction of a nonagon

Distance EF is one side of the nonagon, being a very good approximation of a ninth of the circumference (99.2775%).

8. With radius EF as a chord, draw a circle to cut the original circle and continue this around the circle to divide it into nine parts. Connect these points of intersection to produce the nonagon.

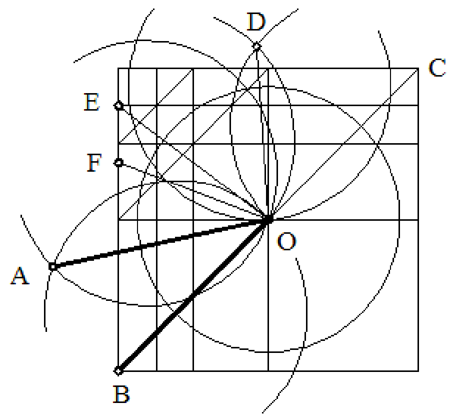
6. CONSTRUCTION OF THE POLYGON WITH ELEVEN SIDES

The hendecagon is not constructible. Bodner (2009) showed how it can be constructed an eleven-pointed polygon with a good approximation [7].

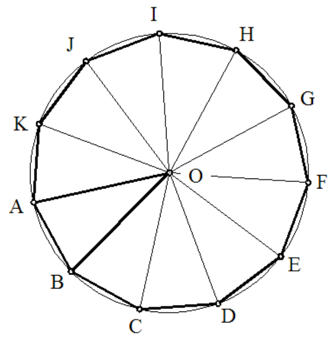
The first construction shown in fig. 10a illustrates how to determine a side of a hendecagon. The internal angle of a circle divided by eleven is 32.7272°.

To inscribe a regular hendecagon in a given circle we must cover the following steps:

1. Given a square with center O, sub-divide the square three times along its diagonal BC to produce a point E on the side of square.
2. Connect the centre of the square with point E. The measure of angle EOC is 98°.
3. Bisect angle EOC to obtain the point D.
4. Bisect angle BOD = 131° to obtain the point F.
5. Bisect angle BOF = 65.5° to obtain the point A. The angle AOB is a very close approximation to 32.7272°. This angle has a value of 32.75°.
6. Repeat this angle around the centre O eleven times consecutively.
7. Superimpose a circle to obtain the points that create the hendecagon (fig. 10b).
8. Connect these points with straight lines.



a)



b)

Fig. 10 Construction of a hendecagon

7. THIRTEEN - POINT GEOMETRY

For dividing a circle into 13 equal parts (construction method is valid for dividing a circle in any equal parts) draw two perpendicular diameters. Vertical diameter is divided into many parts we wish to divide the circle, in this case in 13 parts. Using extremities of vertical diameter (points A and 13) as centers, with a radius equal to the diameter of the circle, describe arcs which intersect it at the points M and N. The points M and N are connecting by turn with points marked with even numbers (2, 4, 6, 8, 10 and 12), extending this lines to intersect the given circle, you have a series of points that divide the circle into 13 equal arcs (fig. 11).

Points M and N can be merged with points marked with odd numbers (1, 3, 5, 7, 9 and 11) and extending this lines to the intersection with the given circle, was obtained another series of points that divide the given circle identically in 13 equal arcs.

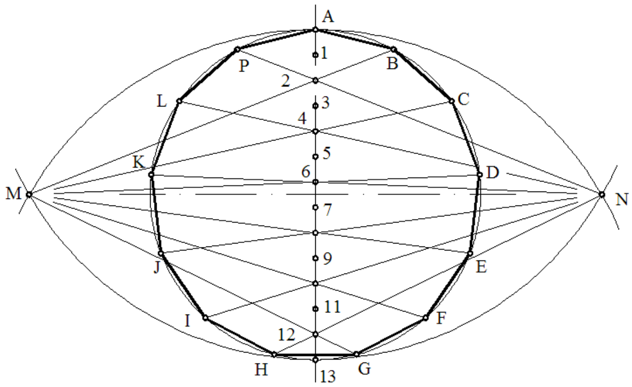


Fig. 11 Construction of a polygon with thirteen sides

8. FIFTEEN - POINT GEOMETRY

The internal angle of a pentadecagon is 24°. It is possible to create 24° angle by combination of the five- and six-point geometries. Figure 12 shows the construction of a pentagon and a hexagon both constructed in the same circle. The angle between the radial lines that connect the center of circle with a vertex of each polygon is 24°.

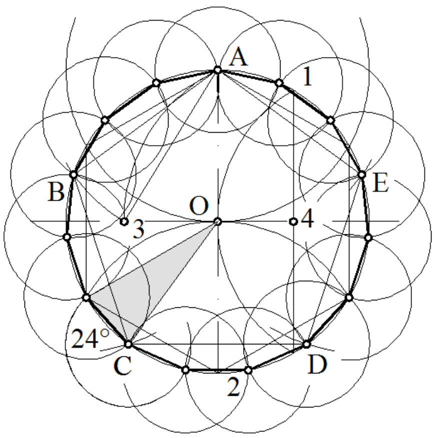


Fig. 12 Construction of a pentadecagon

The points where these radial lines intersect the circle determine one side of the polygon. With this radius in compass as a chord, mark off the remaining points on the circle to step off fifteen divisions along the circumference of the circle. The obtained polygon is known as a pentadecagon, or a quindecagon.

9. GEOMETRIC CONSTRUCTION OF THE POLYGON WITH SEVENTEEN SIDES

The method of construction determines the first and fourth points of the heptadecagon [9]. The internal angle of a circle divided by seventeen is 21.1764°. With a radius in compass equal to the distance between the two points we can locate all seventeen points along the circumference of the circle by moving the compass around it.

To determine the position of the fourth point on the circumference of a given circle proceed as follows:

1. Draw two diameters at right angles to each other.
2. On the vertical radius, OA, locate a point, B, so that OB is a quarter of OA (fig. 13).
3. Connect point 1 and B with straight line.
4. Locate point C so that angle OBC is a quarter of angle OB1.
5. Locate point D so that angle CBD is 45°.
6. With D1 as diameter, draw a semi-circle. This semi-circle will intersect with OA at E.
7. With CE as radius and C as center, draw a semi-circle.
8. Where this semi-circle intersects with O1 at F, raise a perpendicular to intersect with the initial circle at 4.
9. With radius 14 as a chord, locate the remaining points on the circle (fig. 14).
10. Connect these points with straight lines.

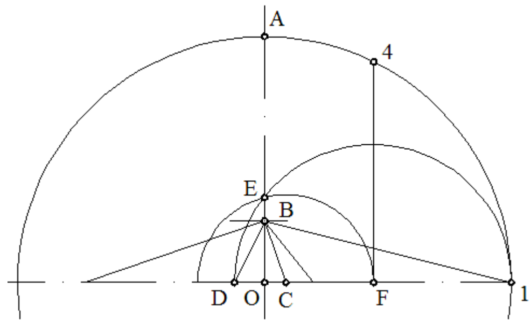


Fig. 13 Discovering the position of the fourth point of a heptadecagon on the circumference of a given circle

Using positions of points 1 and 4 defined above, it is possible to apply the system known as neusis [10] to divide angle 1O4 into three equal parts.

This involve the circle centered at O with radius O4 = O1, and a line which goes through 4, crosses this circle at N and the horizontal line O1 at P, and for which PN = O4 = ON (the same distance as the radius).

The lines PN and ON (which are both equal) form an isosceles triangle with the angle NPO and NOP a third of angle 1O4. Through point N draw a parallel to the horizontal line O1 and determine point 2 at the intersection with the circle centered in O. Connect points 1 and 2 with straight line. Distance 1-2 is one side of the polygon (fig. 15).

With radius 12 as a chord, mark off the remaining points on the circle. Connecting these points with straight lines gives a good approximation of a heptadecagon (differing from L_{17} by less than 0.39%).

10. THE POLYGON WITH NINETEEN SIDES

In fig. 16 is shown a tile which is constructed with nineteen divisions – also known as a nonadecagon or enneadecagon [10]. The internal angle of a circle divided by nineteen is approximately 19° (18.947368°), an angle that is not possible to construct simply by compass and straightedge. However, for the construction of a polygon with 19 sides we can use the method illustrated in fig. 11.

Note that the pattern is symmetrical about its vertical axis but not its horizontal axis.

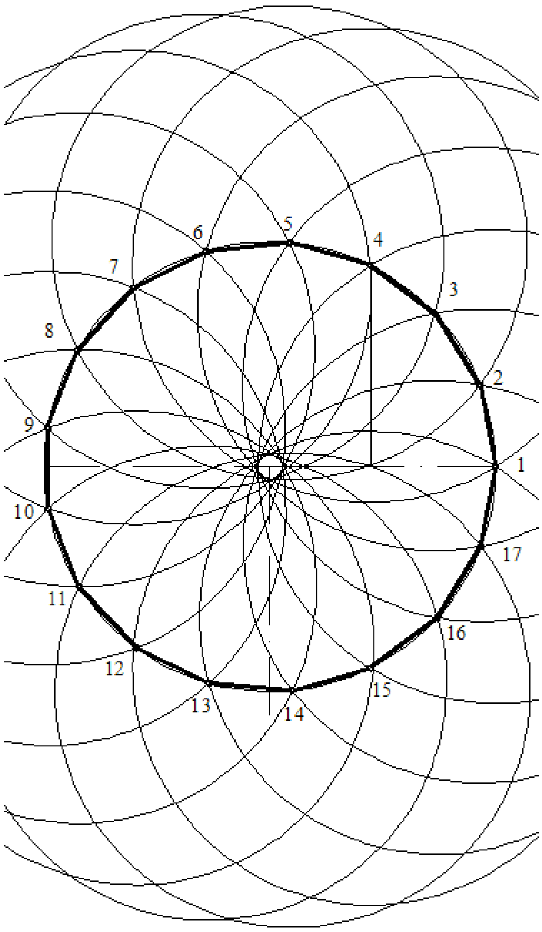


Fig. 14 Construction of a heptadecagon

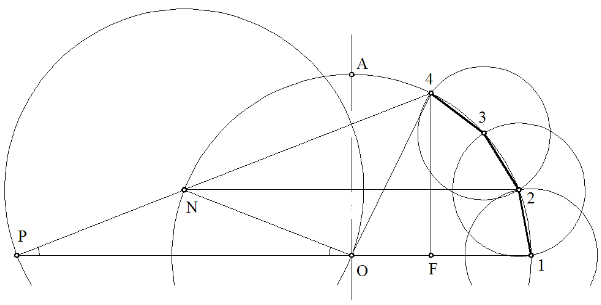


Fig. 15 Discovering the side of a heptadecagon

Fig. 16 Nineteen – point geometry [10]

11. CONCLUSIONS

Geometric constructions are very important to drafters, engineers, architects and designers. Geometric constructions of polygons with any number of sides have important uses, both in solving plane geometry problems and in making artistic or technical drawings.

Any person interested in becoming a good engineer must acquire the necessary informations about the basic principles of Euclidean geometry in solving design problems in a variety of mechanical disciplines such as structural design, industrial design, constructions or CAD applications.

Even if a CAD program allows easy construction of any regular polygon, therefore, everyone in all technical fields needs to know the constructions explained in this paper.

An understanding of the principles of geometric constructions underlying the structure and shape in design can provide high potential as tools for solving design problems nowadays, being beneficial for the development of visual perception of the designer, of his imaginative capabilities and to the perception of forms and their meaning, in general.

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