A Short History of Imaginary Numbers

Misha Nikouravan

Tehran International School (TIS), Tehran, Iran

E-mail addresses: misha1381@gmail.com

ABSTRACT

This paper is discussing how and where imaginary numbers came to be and how their extension to our classic number line helped mathematics to grow even faster. We talk about the beginning of imaginary numbers and the set of rules that come with them. We show how an error that occurred in an equation started the discovery of these. These numbers also help us achieve a better perspective towards the parabolas we see every day. At the end, you can see how these new numbers found the perfect place on the number line and fit in well with different categories we all know.

Keywords: Mathematics, Imaginary numbers

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INTRODUCTION

In mathematics, a real number is a value of a continuous quantity that can represent a distance along a line. Real numbers include all rational numbers, such as integer, fraction, and all irrational numbers. The numbers 1, 2, 3, 4, and etc. are numbers we can easily understand and visualize. For example, two apples or five oranges make perfect sense but you can never visualize what -7 is. Many people had a hard time accepting zero or negative numbers because they would not make sense. However, at some point, people hit some problems where they could not ignore negative numbers anymore. This caused them to extent the number line adding digits before zero. Now days, when we end up with a negative number under a radical with an even index, we often say these problems have no solutions. Therefore, we need to extend our number line again so that we can easily work with different problems. Imaginary numbers have been shaped to make solving these problems easier and smoother. Despite the fact that we might still hit dead-end, but in order to reach to an answer for our problems and any other type of equation, it is needed to complete our number system even more. Imaginary numbers have an interesting history on how they have been shaped, how they are solved and how they have contributed to form complex numbers by the help of Greek mathematician, Heron of Alexandria, who lived sometime between 100 BC

and 100 AD (Roy, 2007). They first appeared in a study concerned with the dimensions of a pyramidal frustum. Although Heron of Alexandria recognized the conceptual possibility of negative numbers possessing square roots, it took a considerable period before they started to become of practical significance. This was owed to discoveries made by Scipione del Ferro and Girolamo Cardano roughly between 1450 and 1600 AD. From 100 AD to the fifteenth century, very little information on imaginary numbers was recorded. Worthy of note are contributions made by scholars such as Diophanrus of Alexandria (circa 300 AD) and Mahaviracarya (circa 850 AD) who both also considered the conceptual possibility of square roots of negative numbers. By the 18th century complex numbers had achieved considerable recognition and were starting to become written as, for example, $3 + 5i$, where 3 represents what is known as the real component and $5i$ is the imaginary component. The letter i here is representative of $\sqrt{-1}$ and first used by Euler in 1777.
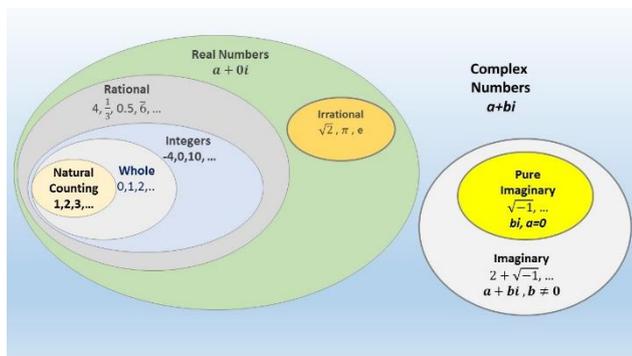

Fig 1

Now we have the same problem with ($\sqrt{-1}$) and that is why mathematicians like Del Ferro, Tartagli and Bombelli came together to solve these. Here we aim to consider a combination of real numbers with non-real numbers that is call imaginary numbers. Mathematicians use the letter "i" to symbolize the square root of -1 ($i = \sqrt{-1}$) and first used by Euler in 1777 as non-real part.

EULER AND DEL FERRO

Cardano was the first to introduce complex numbers $a + \sqrt{-b}$ into algebra, but had misgivings about it (Merino, 2006).
L. Euler (1707-1783) introduced the notation $i = \sqrt{-1}$ (Dunham, 1999), and visualized complex numbers as points with rectangular coordinates, but did not give a satisfactory foundation for complex numbers. The example $x + 5 = 2$ is easy and can be solved now days under seconds but it challenged one of the best mathematicians, Leonhard Euler, back in the 18th century. Euler did not know how to deal with negative numbers and even once said that negative numbers are more than infinite. Negative numbers were ignored on and on because people simply did not know what to do with them. Then, 5 centuries ago, something happened in Europe that did not allow mathematicians to ignore these numbers anymore. Del Ferro, and Italian mathematician, was trying to find a formula for equations with the highest power of 3, cubic ($ax^3 + bx^2 + cx + d = 0$). Since the general term is a little bit more complicated, Del Ferro first considered the case where the x^2 term is missing and the last term is negative ($ax^3 + cx - d = 0$). Since back then, people didn't want to

deal with negative numbers; Del Ferro brought d to the other side and said that c required to be positive ($ax^3 + cx = d$; $d > 0, c > 0$). Then he tried to make x alone by bringing everything else to the other side. It took Del Ferro some clever substitutions but he finally was able to find an equation, where just like the quadratic formula, you need to substituted numbers to find answers:

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} \qquad (1)$$

Del Ferro kept his formula a secret until he was on his deathbed when he finally told his student, Antonio Foir.

FOIR AND TARTAGLIA

After finding out about the formula, Foir challenged a much more skilled mathematician, Fontana Tartaglia. Tartaglia claimed that he could solve cubic equations way before but it was all a lie. At the last minutes before the challenge, he finally found the way to solve them and bet Foir. The reason that Foir's formula did not work well was that sometimes the equation would break under certain c d values. This means that after substituting certain numbers, you would end up with a negative number under the radical with the index of 2 , ($d = 4$, $c = 15$)

$$x^3 = 15\, x + 4 \qquad (2)$$

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} - \frac{c^2}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{c^2}{27}}} \qquad (3)$$

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \qquad (4)$$

CARDAN AND BOMBELLIE

After a famous mathematician, Cardan, found out about the formula and its problem, he came up with a way to solve it. Well we usually say that there is no solutions to our problem when we end up with negative numbers under a radical with an even index but from the way that parabola are formed, we can see that they should always have a solution. As an example, if we have $x^2 + 1$ our graph would be this:
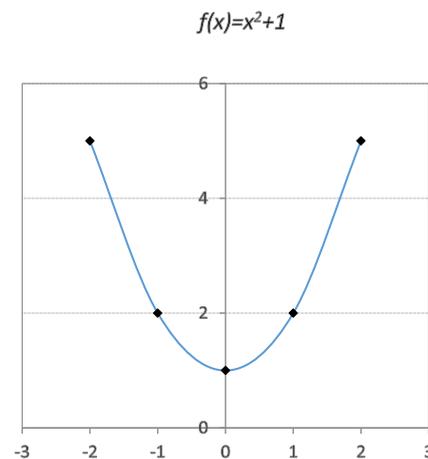

Fig 2

We can obviously see that the graph does not pass the x-axis at all. However, if we look from another perspective, we can see that it actually those cross the x-axis and that we were just looking at it from the wrong angel. This is proof that we should have an answer for our cubic problem despite the fact that our formula ends up with no solutions. Cardan knew this and was thinking for a way to solve the problem (Guilbeau, 1930).

He found a smart way to go around the negative numbers but ended up being stuck in an algebraic loop, where he would start from somewhere but end up at the same place. Unfortunately, Cardan could not solve this problem and passed away. However, his student, Bombelli finally succeeded in solving the problem that was bothering many generations for years. Bombelli first accepted the fact that if both positive and negative numbers would not work, then there should be another type of numbers that did work. Then, rather than dreaming about a new number or a new symbol, Bombelli simply said let $\sqrt{-1}$ be a number itself.

In the past, everyone agreed and said that the square root of negative numbers cannot exist, but Bombelli simply accepted them and said that they may exist. Although $\sqrt{-1}$ was said to be a new number for itself, Bombelli could not find a place for the new number on our number line. That led to the question that if $\sqrt{-1}$ is even a "real" number. Bombelli saw that the number line had been improved and extended many times before and that this needs to be done again so that we can fit our new number in. Before this, Bombelli tried to simplify Cardan's equation

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} \qquad (5)$$

by saying "the root of minus 1 parts of each half of the equation must cancel out when added together" (a and b are constants we need to find)

$$\left(\sqrt[3]{2 + \sqrt{-121}}\right)^3 = \left(a + b\sqrt{-1}\right)^3$$
$$\left(\sqrt[3]{2 - \sqrt{-121}}\right)^3 = \left(a - b\sqrt{-1}\right)^3 \qquad (6)$$

First, he eliminated the cube root, which then resulted in a system of equation

$$2 = a(a^2 - 3b^2) \qquad (7)$$

$$11 = b(3a^2 - b^2) \qquad (8)$$

However, Bombelli got around this by some checking and guessing. He took the original equation and substituted a few numbers and finally found that 4 is the solution. So now if we substitute 4 into our new equation we will get $a = 2; b = 1$. If you replace these numbers, we will see that the answer is 4. We have found the solution to Cardan's problem! The interesting this is that nor our solution or our process contained ($\sqrt{-1}$), but we found out that by extending our number system to contain, $\sqrt{-1}$, we can solve equations like this.

Now that we know these numbers can help us solve equations that were once considered "impossible", we need to know their nature and where they stand on the complex plane. $\sqrt{-1}$ Was at first considered as a "hack" for solving some mathematical

equations. After all, just like 0 and negative numbers, no one could imagine what $\sqrt{-1}$ was in the real world. That is the reason they have given the terrible name *imaginary*. $\sqrt{-1}$ Is actually a number, like all the other numbers, has a place on the complex plane, and even has a pattern. After a century later, Euler started using i as a symbol instead of $\sqrt{-1}$ so that he would avoid writing this number on and on. When $\sqrt{-1}$ was named imaginary, in response, everything else on the number line took the name of "real". When we then put together a real and an imaginary number, we get what we call a Complex Number (ex 3+2i). All of complex analysis can be developed in terms of ordered pairs of numbers, variables, and functions $u(x,y)$ and $v(x,y)$ (Arfken & Weber). Although we know about i, we cannot find a place for it on the number line. Now remember our original problem? We wanted to find a number that when multiplied by it self, lead us to a negative number. Well when we take 3, as an example, we can see that (3×3) is 9 (a positive number) and $\{(-3) \times (-3)\}$ is again 9, another positive number. From this, we can understand that no matter where we start on the real number line, we would always end up turning 180° back into the positive side of the number line:
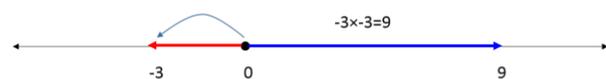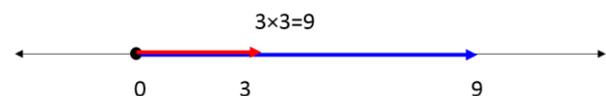
Fig 3

Fig 4

Therefore, what we need is a number, to which when multiplied by itself, would turn 90° not 180°. This is what imaginary numbers do. Therefore, from this, we can create the complex number line, where imaginary number is at a right angle to our number line. When we raise a real numbers to a power, they get bigger and bigger as they naturally should. Nevertheless, when we raise i to a higher power, the number does not increase. Instead, it creates a pattern after every 4 multiplications:

$$i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$$

$$i^5 = i \quad i^6 = -1 \quad i^7 = -i \quad i^8 = 1$$

(9)

So when we take 1, as an example, and multiply it by i, we get i, and when we multiple i with i again, we get -1. This is the 90° rotation we were looking for. When we keep multiplying with i, we keep rotating around the plane and getting the same numbers every 4 multiplication. So we cannot say that imaginary numbers are apart from our number line or just a random hack, they are in fact an extension to our number line

but can be seen when we look at numbers from 2 dimensions. Sometimes, it is not obvious that we are missing a set of number.

We can figure if we are by using the mathematical idea of closure. What we need to do is to find out under which operations different sets of numbers close under, and if a set results to another number, which is not, included in our set, that is when we need to add another part of numbers to math.
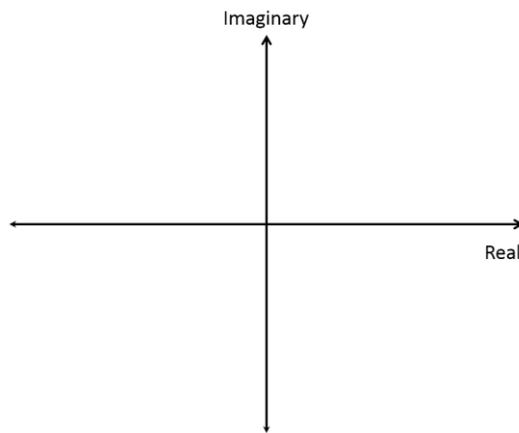

Fig 5

CONCLUSION

A real number is a value that represents a quantity along a continuous number line. Real numbers can be ordered. The symbol for the set of real numbers is script R. The real numbers include: counting natural numbers $N = \{1, 2, 3, \dots\}$, Whole numbers $\{0, 1, 2, 3, \dots\}$, Integers $Z = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, Rational numbers such as $Q = (-\frac{1}{2}, 6.25, 0.6\overline{25}, \sqrt{3})$ and Irrational numbers such as $\{0.121221222\dots, \sqrt{3}, \pi\}$. The basic properties of real numbers are used to determine the order in which we can simplify mathematics expressions. There are four mathematical properties, which involve addition. The properties are the commutative, associative, additive identity and distributive properties. Real numbers are closed under addition, subtraction, and multiplication. The commutative property says that the positions of the numbers in a mathematical equation do not affect the ultimate solution. Five plus three is the same as three plus five. This applies to addition, regardless of how many numbers you add together. The commutative property allows you to add a large group of numbers together in any order.

The commutative property does not apply to subtraction. Five minus three is not the same as three minus five. If we try subtraction for natural numbers, we can see that for some natural numbers (such as $6 - 4 = 2$) we get a natural number as an answer, but for ex. 4-6, we will not end up with a natural number anymore. We do not have any negative numbers or zero in outer set. Therefore, we need to expand out number set to include negative numbers and zero. These are called integers. After we expanded our number set to include integers, we could see that although natural numbers are not closed under subtraction, integers are (since subtracting any integer from the other will result to another integer).

As we use more mathematical operations, we have to expand our number set even more in order to include answers for these operations. As we bring division, we learn to include fractions in our number set.

That is when rational numbers come in. From this, we can say that all integers are rational numbers but not all rational numbers are integers. Now we can see that rational numbers are closed under addition, subtraction, multiplication and division. However, not under root or power since sometimes raising a rational number to a power will not result in another rational number. It turns out that $\sqrt{2}$ is not resulting in a rational number since there are no two EXACT numbers that when multiplied together result in 2. Moreover, because of this, we give the name "irrational" to these numbers. There is even another group of numbers that are named Transcendental (which are like e or π...).

Therefore, as we include irrational numbers, we now have what we call real numbers. After including all these different types of numbers, do we still result in a real number when we square another real number? Well, despite all these extensions, we might still have something missing from our number set. As an example, the square root of -9 has no solution and because of that problem, we have to expand our number system even more to include imaginary numbers ($\sqrt{-9} = 3i$). Taking all our numbers, from real to imaginary, we finally reach to what we call complex numbers. Despite this, some mathematicians say that even complex numbers are not sufficient and that problems like $\sqrt{-i}$ will not result in a complex number. However, that turned out to not be the case. We can in fact evaluate the square root of $-i$ using the complex plane. "Since $-i$ has a magnitude of 1 at an angle of -90 degrees; we just need a number with a magnitude of 1 at an angle of -45 degrees. According to our unit circle,

$$\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right) i \qquad (10)$$

Therefore, the square root of -1 is just another complex number. In fact, there is no operation that the complex numbers cannot handle.

Table 1

Numbers	Symbol	Examples	Closed under
Natural	$\mathbb{N}$	1, 2, 3, 4, …	$+, \times$
Integers	$\mathbb{z}$	-2, -1, 0, 1, …	$+, -, \times$
Rational	$\mathbb{Q}$	1/2, 0.7, 2, …	$+, -, \times, \div$
Real	$\mathbb{R}$	-1/2, π, $\sqrt{2}$, 1, …	$+, -, \times, \div$
Complex	$\mathbb{C}$	1+2i, 2i, 2…	$+, -, \times, \div, \sqrt{}, ()^2$

In conclusion, we can understand that by expanding our number system we can easily solve many mathematical equations and problems and although many problems might seem like you hit a dead end, there's always a way to solve them. We just have to learn to accept some "impossible numbers" because they make algebra complete.

Without imaginary numbers, we would have still been stuck on Cardan's problem for who knows how many years. We also understood that by having a new perspective on math and numbers in general, we are able to do almost anything.

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